Moreover, since $|x_k - jx_{k+1}|^2 = x_k^2 + x_kx_{k+1} + x_{k+1}^2$ we conclude that

$$\|Z\|^2 = \sum_{k=1}^{n}(x_k^2 + x_kx_{k+1} + x_{k+1}^2)$$

Therefore, the Cauchy-Schwarz inequality $|\langle 1, Z\rangle|^2 \leq \|1\|^2 \, \|Z\|^2$ is equivalent to the lower inequality.
The upper inequality is easier, since $\langle X, Y\rangle \leq \|X\| \, \|Y\| = \|X\|^2$, so

$$\sum_{k=1}^{n}(x_k^2 + x_kx_{k+1} + x_{k+1}^2) = \|X\|^2 + \langle X, Y\rangle + \|Y\|^2 \leq 3\|X\|^2$$

and the is equivalent to the upper inequality.
Remark. We only need the fact that the x_k's are real. The positivity assumption is unnecessary.

Solution 2 by Arkady Alt, San Jose, California, USA. For cyclic sum $\sum\limits_{k=1\ cyclic}^{n} \dfrac{x_k^2 + x_kx_{k+1} + x_{k+1}^2}{3}$ we will use more compact notation $\sum\limits_{cyc}^{n} \dfrac{x_1^2 + x_1x_2 + x_2^2}{3}$.
Noting that $\left(\dfrac{x+y}{2}\right)^2 \leq \dfrac{x^2+xy+y^2}{3} \iff 0 \leq (x-y)^2$ (for any real x, y) we obtain $\dfrac{1}{n}\sum\limits_{cyc}^{n} \dfrac{x_1^2 + x_1x_2 + x_2^2}{3} \geq \dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2$. By Quadratic Mean-Arithmetic Mean Inequality $\dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2 \geq \left(\dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)\right)^2 = \left(\dfrac{\sum_{k=1}^{n} x_k}{n}\right)^2$.
(Or, applying Cauchy Inequality to $\left(\dfrac{x_1+x_2}{2}, \dfrac{x_2+x_3}{2}, ..., \dfrac{x_n+x_1}{2}\right)$ and $(1, 1, ..., 1$ we obtain $\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2 n \geq \left(\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)\cdot 1\right)^2 \iff \dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2 \geq \left(\dfrac{1}{n}\sum\limits_{cyc}^{n}\dfrac{x_1+x_2}{2}\right)^2 = \left(\dfrac{\sum_{k=1}^{n} x_k}{n}\right)^2$. Since $\dfrac{x^2+xy+y^2}{3} \leq \dfrac{x^2+y^2}{2} \iff 0 \leq (x-y)^2$ (for any real x, y) we obtain $\dfrac{1}{n}\sum\limits_{cyc}^{n}\dfrac{x_1^2+x_1x_2+x_2^2}{3} \leq \dfrac{1}{n}\sum\limits_{cyc}^{n}\dfrac{x_1^2+x_2^2}{2} = \dfrac{\sum_{k=1}^{n} x_k^2}{n}$.

Solution 3 by Michel Bataille, Rouen, France. Since $x_kx_{k+1} \leq \frac{x_k^2+x_{k+1}^2}{2}$, we have $x_k^2 + x_kx_{k+1} + x_{k+1}^2 \leq \frac{3}{2}(x_k^2 + x_{k+1}^2)$. It follows that

$$\frac{1}{n}\sum_{k=1\text{ cyclic}}^{n}\frac{x_k^2 + x_kx_{k+1} + x_{k+1}^2}{3} \leq \frac{1}{n}\sum_{k=1\text{ cyclic}}^{n}\frac{1}{2}(x_k^2 + x_{k+1}^2) = \frac{1}{n}\sum_{k=1}^{n}x_k^2,$$